Enrollment No: ____________________ **Exam Seat No:**____________________

C. U. SHAH UNIVERSITY

Winter Examination-2022

Subject Name : Theories of Ring and Field

Subject Code : 5SC03TRF1 **Branch: M.Sc. (Mathematics)**

Semester: 3 **Date: 24/11/2022** **Time: 11:00 To 02:00** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1		**Attempt the Following questions**	**(07)**
	a.	Define: Ring.	(1)
	b.	Define: Monic Polynomial.	(1)
	c.	Give an example of division ring which is not field.	(1)
	d.	How many ideals are in field?	(1)
	e.	Define: Simple ring.	(1)
	f.	Define: Radical of the ring.	(1)
	g.	Define: Skew Field.	(1)
Q-2		**Attempt all questions**	**(14)**
	A	If $\boldsymbol{R}$ is ring such that $\boldsymbol{a}^2 = \boldsymbol{a}$, $\forall \boldsymbol{a} \in \boldsymbol{R}$ prove that R is a commutative ring.	(05)
	B	Prove that every field is an integral domain but converse is not true.	(05)
	C	Show that the set $\boldsymbol{I} = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \middle\vert \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{R} \right\}$ is a subring of $\boldsymbol{M}_2(\boldsymbol{R})$.	(04)
		OR	
Q-2		**Attempt all questions**	**(14)**
	A	State and prove Division algorithm for polynomial ring.	(07)
	B	Define Field. Show that a field has no proper ideal. The converse is true always?	(05)
	C	Define Unique factorization domain.	(02)
Q-3		**Attempt all questions**	**(14)**
	A	State Eisenstein criterion theorem and check whether which of the following polynomial are irreducible over $\boldsymbol{Q}$ or not (i) $f(x) = 21x^3 - 3x^2 + 2x + 9$ (ii) $x^5 + 9x^4 + 12x^2 + 6$.	(07)
	B	Let $\boldsymbol{R}$ be a commutative ring and let $\boldsymbol{a} \in \boldsymbol{R}$. Then show that the set	(03)

$I = \{x \in R \ /ax = 0\}$ is an ideal of R.

C Show that the set $I = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a, b \in Z\right\}$ is not an ideal of $(M_2(Z), +, .)$. (04)

OR

Q-3 A Let I_1 and I_2 be any ideals of a ring R. Then prove that $I_1 \cup I_2$ is an ideal of R iff either $I_1 \subset I_2$ or $I_2 \subset I_1$. (06)

B $(R, +, \cdot)$ is ring then prove that for $a, b \in R$. (04)

(i) $a0 = 0a = 0$ (ii) $a(-b) = -(ab) = (-a)b$

C Check whether the polynomial $x^3 + 2x + 2$ is reducible over Z_7 or not? (04)

SECTION – II

Q-4 **Attempt the Following questions** **(07)**

a. Is a Polynomial ring always a field? (01)
b. State: Division algorithm for ring polynomial. (01)
c. Define: Irreducible polynomial over integral domain D. (01)
d. The characteristic of ring $(Z_{2022}, +_{2022}, \times_{2022})$ *is*______. (01)
e. Define: Maximal ideal for ring. (01)
f. Define: Algebraic extension over a field. (01)
g. Define: Primitive polynomial (01)

Q-5 **Attempt all questions** **(14)**

A State and prove first isomorphism theorem for rings. (07)

B Define Integral domain and prove that every finite integral domain is field. (07)

OR

Q-5

A Let R be a commutative ring with unity and let $S \neq R$ be an Ideal of R Then prove that R/S is an Integral domain $\Leftrightarrow$ S is prime Ideal of R. (06)

B The principal ideal generated by two element$a$ and b of an integral domain are equal if and only a and b are associate. (05)

C Let D be an Integral domain a two non zero element $a, b \in D$ are associate if and only if a/b and b/a. (03)

Q-6 **Attempt all questions** **(14)**

A If $f(x)$ & $g(x)$ are two polynomials in $F[x]$; F is a field or an integral domain then prove that $\deg(f(x)g(x)) = deg f(x) + deg\ g(x)$. Does the result holds if F is a commutative ring? Justify. (07)

B Show that $(Z, +, .)$ is an Euclidean ring. (04)

C $\mathbb{K}$ be the extension field of $\mathbb{F}$ let $a \in \mathbb{K}$ be algebraic over $\mathbb{F}$, let $p(x)$ be a minimal polynomial for a over $\mathbb{F}$ then $p(x)$ is irreducible over $\mathbb{F}$. (03)

OR

Q-6 **Attempt all Questions**

A Define kernel of ring homomorphism from ring $\boldsymbol{R}$ to $\boldsymbol{R'}$ and prove that $\boldsymbol{ker}\emptyset = \{\boldsymbol{a} \in \boldsymbol{R} | \emptyset(\boldsymbol{a}) = \boldsymbol{0}\}$ is subring of $\boldsymbol{R}$ (04)

B Let $\boldsymbol{F}$ be a field and $\boldsymbol{p}(\boldsymbol{x}) \in \boldsymbol{F}(\boldsymbol{x})$, then $< \boldsymbol{p}(\boldsymbol{x}) >$ is maximal ideal in$\boldsymbol{F}[\boldsymbol{x}]$ if and only if $\boldsymbol{p}(\boldsymbol{x})$ is irreducible over$\boldsymbol{F}$. (06)

C Define; algebraic element and prove that $\mathbb{K}$ be the extension field of $\mathbb{F}$ let $\boldsymbol{a} \in \mathbb{K}$ be algebraic over $\mathbb{F}$ then any two minimal polynomial for $\boldsymbol{\alpha}$ over $\mathbb{F}$ are equal. (04)